Title: Derived algebraic geometry + representation theory I

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Disclaimer: In my talk there will be no dag, rep.th., only algebra + topology

<u>Outline</u>: 1. Def'ns: affine Hecke algebra $\mathbb{H}$, Steinberg variety St

2. Statement of the main theorem and proof for SL_2

$$\mathbb{H} \simeq K^{G\times \mathbb{C}^*}(St)$$

3. (?) Relation with p-adic groups and appearance of dg categories, derived fiber product

<u>Reference</u>: Section 7.5 in Chriss-Ginzburg "Representation theory and Complex Algebraic Geometry"

G semisimple simply connected cplx algebr. gp ($G = SL_n, SL_2$)

G group, T max'l torus   <u>root system</u> $R \subset P = \mathrm{Hom}(T, \mathbb{C}^*)$ weight lattice

$\alpha \in P$ $\alpha^\vee \in P^\vee = \mathrm{Hom}(P, \mathbb{Z})$, $\langle \alpha, \alpha^\vee \rangle = 2$

st: $s_\alpha : P \to P$ simple reflections

$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ st. ....?

$R = R^+ \sqcup R^-$ simple roots: positive roots are lin comb of pos. roots (pos.)

<u>Weyl gp</u> W generated by s_α's, $\alpha \in R$

<u>Ex</u>: simple roots of SL_n: $\varepsilon_i - \varepsilon_{i+1}$

<u>Hecke algebra</u> H_W q-deformation of gp ring of Weyl group $\mathbb{C}[W]$

Length function on W: $w = s_{\alpha_1} \cdots s_{\alpha_n}$ $\ell(w) = n$ is well-def'd

generators of H_W over $\mathbb{Z}[q, q^{-1}]$: $T_w : w \in W$

relations:

(a) $(T_s + 1)(T_s - q) = 0$,   s simple reflection

(b) $T_y T_w = T_{yw}$ if $\ell(y) + \ell(w) = \ell(yw)$

affine Weyl group $\quad W_{aff} = W \ltimes \mathbb{C}[P]$

think of as ...
reflections $\qquad$ translations

affine Hecke algebra $\quad$ free $\mathbb{Z}[q,q^{-1}]$-module with basis
$\{e^\lambda \cdot T_w \mid w \in W, \lambda \in P\}$

for $\lambda \in P$, write $e^\lambda \in \mathbb{Z}[P]$ for element corresponding to it.

st.

(a) $\{T_w\}$ span a subalgebra of $\mathbb{H}$ isomorphic to H_W

(b) $\{e^\lambda\}$ span a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathbb{H}$ isomorphic to $R(T)[q,q^{-1}]$ where $R(T) = \mathbb{C}[\mathrm{Hom}(T, \mathbb{C}^*)]$

(c) For $s = s_\alpha \in S$ with $\langle \lambda, \alpha_s^\vee \rangle = 0$

$$T_s e^\lambda = e^\lambda T_s$$

(d) for $s = s_\alpha \in S$ with $\langle \lambda, \alpha_s^\vee \rangle = 1$

$$T_s e^{s(\lambda)} T_s = q e^\lambda$$

"Topology": Steinberg variety St

Def'n: $\mathcal{B} = G/B_0$ flag variety $\qquad B_0 \subset G$ Borel subgroup

$= \{\mathfrak{b}$ Borel subalg. in $\mathfrak{g}\}$

Ex: $B_0 \subset SL_n$ upper triangular matrices

$G = SL(V)$ is variety of flags

$\{V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V\}$

$\{$nilpotent elts of $\mathfrak{g}\} =: \mathcal{N}$

$\mathcal{N} = T^*\mathcal{B} = G \times_{B_0} \mathfrak{b}_0^\perp = G \times_B \mathfrak{n}$ $\quad$ nilpot. elts of $\mathfrak{b}_0$

$= \{(n, \mathfrak{b}) : n$ nilpotent matrix, $\mathfrak{b}$ Borel subalgebra in $\mathfrak{g}$, $n \in \mathfrak{b}\}$

$G = SL_n$
$\mathfrak{g} = \mathfrak{sl}_n$

$$St = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \qquad \begin{array}{ccc} \tilde{\mathcal{N}} & = \{(n,b) : n \text{ nil } b,\ b \text{ Borel in } \mathfrak{g}\} \\ \downarrow & \downarrow \\ \mathcal{N} & \ni n \end{array}$$

$$= \{(n, b_1, b_2) : n \in b_1 \cap b_2\} \qquad \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} = T^*\mathcal{B} \times T^*\mathcal{B}$$

b_1, b_2 Borel s/alg. of $\mathfrak{g}$ $\qquad \cong T^*(\mathcal{B} \times \mathcal{B})$

<u>Prop:</u> St = union of conormal bundles to G-orbits in $\mathcal{B} \times \mathcal{B}$

<u>Convolution in equivariant K-theory</u>

Let M_1, M_2, M_3 be smooth quasi-projective G-varieties

$$p_{ij} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$$

choose $Z_{12} \subset M_1 \times M_2$ $\quad Z_{23} \subset M_2 \times M_3$ $\quad$ G-stable closed subvarieties st

$$\begin{tikzcd} & M_1 \times M_2 \times M_3 \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ M_1 \times M_2 & M_1 \times M_3 & M_2 \times M_3 \end{tikzcd}$$

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is a <u>proper</u> map.

Let $Z_{12} \circ Z_{23} :=$ image of p_{13}.

$\mathcal{F}_{12} \in K^G(Z_{12})$ $\qquad$ define <u>convolution</u>:

$\mathcal{F}_{23} \in K^G(Z_{23})$

$$\mathcal{F}_{12} * \mathcal{F}_{23} := (p_{13})_* \left(p_{12}^* \mathcal{F}_{12} \otimes p_{23}^* \mathcal{F}_{23} \right) \in K^G(Z_{12} \circ Z_{23})$$

$$* : K^G(Z_{12}) \otimes K^G(Z_{23}) \longrightarrow K^G(Z_{12} \circ Z_{23})$$

<u>Now:</u> Take $M_i := \tilde{\mathcal{N}}$, $Z_{12} = Z_{23} = Z_{13} = St$. $\qquad G \times \mathbb{C}^* \circlearrowright St$ action ~ %

Then $Z_{12} \circ Z_{23} = St$, use this to get convolution

$$* : K^{G\times\mathbb{C}^*}(St) \times K^{G\times\mathbb{C}^*}(St) \longrightarrow K^{G\times\mathbb{C}^*}(St) \quad \text{associative algebra}$$

Ex: $G = SL_n$ $\quad SL_n \circlearrowright \mathcal{B} = SL_n / B_0 = \{\text{Borel subalgebras } \mathfrak{b} \subset \mathfrak{g}\}$

$$g \in SL_n, \quad \mathfrak{b} \xmapsto{g\cdot} g\mathfrak{b}g^{-1}$$

$$\rightsquigarrow \left.\begin{array}{l} SL_n \circlearrowright T^*\mathcal{B} = \tilde{\mathcal{N}} \\ \mathbb{C}^* \circlearrowright T^*\mathcal{B} \text{ by rescaling fibers} \end{array}\right\} \Rightarrow SL_n \times \mathbb{C}^* \circlearrowright \underset{\tilde{\mathcal{N}}}{\underset{\shortparallel}{T^*\mathcal{B}}}$$

$$\Longrightarrow SL_n \times \mathbb{C}^* \circlearrowright \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = St.$$

Remark: $K^G(X) := K_0(Coh^G(X))$

Theorem (Kazhdan-Lusztig, Ginzburg)

$$\mathbb{H} \cong K^{G \times \mathbb{C}^*}(St_G)$$

Remark: There is a categorified version of this, but there convolution is not associative anymore
$\rightsquigarrow$ need dg categories and derived algebraic geometry

Sketch of proof for $G = SL_2$:

$G = SL_2 \quad G/B_0 = \mathbb{P}^1 = \mathbb{C}\mathbb{P}^1 \qquad B_0 = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2\right\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a:c)$$

Recall prop: $\underset{T^*(\mathbb{P}^1 \times \mathbb{P}^1)}{\underset{\cap}{St}}$ = union of conormal bundles to G-orbits in $\mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft SL_2$

There are 2 such orbits $\Rightarrow St = \underset{St_\Delta}{\overset{\shortparallel}{T^*_\Delta(\mathbb{P}^1 \times \mathbb{P}^1)}} \cup \underset{St'}{\overset{\shortparallel}{T^*_{(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta)}(\mathbb{P}^1 \times \mathbb{P}^1)}}$

conormal bundle for $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$

$X \subset Y \quad \rightsquigarrow TX \subset TY$

normal bundle

$$0 \to TX \to TY|_X \to N_Y X \to 0 \quad \Rightarrow T_Y^* X := (N_Y X)^*$$

v.b. on X

dimension $= 2$ $\left(\begin{array}{l} \text{b/c components} \\ \text{are Lagrangian in } T^*(\mathbb{P}^1\times\mathbb{P}^1)\end{array}\right)$

Define a map

$$\theta : \mathbb{H} \longrightarrow K^{SL_2\times\mathbb{C}^*}(St)$$

by defining image of generators.

Interesting part: verifying relations!

$\mathbb{H}$: associative $\mathbb{C}[q,q^{-1}]$ algebra on 3 generators T, X, X^{-1} st

·) $(T+1)(T-q) = 0$

·) $X\cdot X^{-1} = X^{-1}\cdot X = 1$

·) $T\cdot X^{-1} - X\cdot T = (1-q)X$

Here T is generator for $\mathbb{H}_\omega \subset \mathbb{H}$

X — " — of weight lattice

Write $c := -(T+1) \in \mathbb{H}$

$\theta(X) := [\mathcal{O}_{-1}]$

$\theta(X^{-1}) := [\mathcal{O}_1]$

$\theta(c) := [q\,Q]$

Use: SL_2-equivariant line bundles on $\mathbb{P}^1$ are $\mathcal{O}(n)$, $n\in\mathbb{Z}$

$\mathbb{P}^1\times\mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1$

Q: on the fibers $\simeq \mathbb{P}^1$

$\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1}$ on $\mathbb{P}^1\times\mathbb{P}^1$

$\boxed{Q := \pi^*\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1/\mathbb{P}^1}}$

$\pi : St' \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$

$\cap$

$T^*\mathbb{P}^1\times T^*\mathbb{P}^1$ zero section

$St_\Delta \xrightarrow{\pi_\Delta} \mathbb{P}^1_\Delta = \mathbb{P}^1$

$\cap$

$\mathbb{P}^1\times\mathbb{P}^1$ diagonal

$\boxed{\mathcal{O}_n := \pi_\Delta^*\mathcal{O}(n)}$ on St_Δ

Take $\mathcal{O}_n$ on St supported on St_Δ